5.8. Quantifier Semantics Revisited: Instances

1. Instances. We have so far examined only instances of very simple quantified sentences, such as "$\forall x\, Gx$" and "$\exists y \sim Gy$," where the scope formula is an atom or negation of an atom. But the full details of sentence instances, holding even for more complex scope formulas, requires more attention to detail than those simple cases did. We impose three conditions on sentences qualifying as an instance of a quantified sentence – each condition needed to ward off logical missteps.

Previously an instance of a quantified sentence was constructed by (i) removing the quantifier from the left of that sentence, then (ii) replacing the variable with a name letter in the scope formula that remains. For example, with "$\exists x\, Gx$" we remove "$\exists x$," leaving just the scope formula "Gx"; and then replace the "x" in "Gx" with a name letter – yielding, say, "GA" or "GB".

Scope Formula:	**Instances of This Formula**
Gx	GA
	GB
	GC
	(etc.)

We will impose the following three conditions on the second part of that procedure, where names replaces variables.

(A) Replace **all free** occurrences of the variable with name letters.
(B) Replace **only free** occurrences of that variable with name letters.
(C) Replace all free occurrences of that variable with the **same** name letter.

The reason for imposing condition **(A)** is obvious: in a given model, we want the truth or falsehood of a quantified sentence to depend on the truth or falsehood of its instances. But since only sentences (not quasi-sentences) are capable of being true or false, an instance of a quantified sentence had better contain **no** free variables. Were an instance to retain any free occurrence of a variable, it would fail in its role as a semantic stand-in for the scope formula.

To understand condition **(B)** – that only **free** occurrences of a variable be replaced with name letters – consider the following existential claim.

For some object, x: that object's a cat, but there's something that isn't a cat.

(1) $\exists x\,(Gx \wedge \exists x \sim Gx)$

So consider the following model, with two objects: Neko, who is a cat, and Rex, who isn't.

A: Neko　　　**G__**: is a cat
B: Rex

D: {**Neko**, **Rex**}

A: **Neko**　　　**G**: {**Neko**}
B: **Rex**

Condition (B) has us construct the following instances of sentence (1).

(2) $(GA \wedge \exists x \sim Gx)$
(3) $(GB \wedge \exists x \sim Gx)$

And we expect existential sentence (1) to be true in this model if it has **at least one true instance**.

Now in fact (2) is true in this model. Since Rex isn't a cat here, clearly the right part of (2), "$\exists x \sim Gx$," is true. And since Neko is a cat, the left part of (2), "GA," is true as well. With both parts of the conjunction true, (2) as a whole is true. And since (2) is a true instance of (1), **(1) is true** here as well.

That makes intuitive sense: in a situation where Neko's a cat and Rex isn't, it will be true that there's an object (namely, Neko) such that: that object is a cat, but something isn't. **Treating (2) and (3) as instances of (1) yields the right result**.

By contrast, if we disregard Condition (B), and replace even the bound variables in the scope formula of (1), the consequences are less happy. For in that case the instances of (1) would be (4) and (5).

☠ **Instances of "$\exists x\,(Gx \wedge \exists x \sim Gx)$" ?** ☠

(4) $(GA \wedge \exists x \sim GA)$
(5) $(GB \wedge \exists x \sim GB)$

In order to evaluate the truth of (4) and (5), note first that the right half of each contains a **vacuous quantifier**. The "$\exists x$" in "$\exists x \sim GA$" is vacuous because that quantifier has no occurrences of "x" to bind in its scope formula "$\sim GA$". Now as noted earlier[1], a vacuous quantifier is semantically empty – making "$\exists x \sim GA$" semantically equivalent to "$\sim GA$". The same holds for the right half of (5): "$\exists x \sim GB$" is semantically equivalent to just "$\sim GB$".

That means that (4) and (5) are semantically equivalent to (6) and (7), respectively.

(6) $(GA \wedge \sim GA)$
(7) $(GB \wedge \sim GB)$

Both these quantifiers are contradictions, and are **not true in any model**.

So: **violating condition (B)**, and replacing even bound occurrences of a variable, **yields the wrong results**.

The final restriction is that we replace the variable of quantification by the same name letter throughout.

(C) Replace all free occurrences of that variable with the **same** name letter.

Whereas (1) didn't look like a contradiction, Sentence (8) does look like one: intuitively, there should be no possible way (8) could be true.

(8) $\exists x\,(Gx \wedge \sim Gx)$

[1] In Section 1 of "5.6. Construction Revisited: Quantifiers, Variables, and Binding".

To construct an instance of (8), we remove the quantifier "∃x", and replace every free occurrence of "x" in the remaining scope formula "(Gx ∧ ~Gx)" with a name letter. Now if, following Condition **(C)**, we use the same name letter throughout, we end up with an instance of the following sort.

(GA ∧ ~GA)
(GB ∧ ~GB)
(GC ∧ ~GC)
(etc.)

Since all of these are contradictions, true in no model, the same will hold for (8) – **the correct result**.

But suppose we disregard Condition **(C)**, and replace different occurrences of "x" in "(Gx ∧ ~Gx)" with different name letters. Then Sentence (9) would be treated as an instance of (12).

☠ Instance of "∃x (Gx ∧ ~Gx)" ? ☠

(9) (GA ∧ ~GB)

(9) can certainly be true in a model. In fact the same model as before will suffice: a model with just two objects, cat Neko and non-cat Rex.

(8) ∃x (Gx ∧ ~Gx)

(9) (GA ∧ ~GB)

G__: is a cat

D: {**Neko**, **Rex**}

A: **Neko** **G**: {**Neko**}
B: **Rex**

If (9) counts as an instance of (8), then (8) has at least one true instance in a model, and so isn't a contradiction after all. That seems like **the wrong result**. Hence the appeal of Condition (C): when replacing free occurrences of a variable, we use the same name letter throughout.

2. Quantifier Semantics Revisited. With the official version of "instance" in hand, the semantics for quantified sentences is straightforward.

> An **existential** sentence is **true** in a model if (and only if) it has **at least one true instance** in that model.
>
> An **existential** sentence is **false** in a model if (and only if) **every instance** of that sentence is **false** in the model.
>
> A **universal sentence** is **true** in a model if (and only if) **every instance** of that sentence is **true** in the model.
>
> A **universal sentence** is **false** in a model if (and only if) it has **at least one false instance** in that model.

So the following model assigns the truth value listed for each of these quantified sentences.

D: {**2**, **3, 4**}

A: 2	**G**: {4}
B: 3	**H**: {3, 4}
C: 4	**I**: {2, 3, 4}
	J: { }

$(GA \rightarrow \exists x\, Gx)$: **1**
$(GC \rightarrow \exists x\, Gx)$: **1**
$\forall x\, Hx$: **0**
$\forall x\, (Ix \rightarrow Hx)$: **1**
$\exists x\, (Jx \rightarrow Gx)$: **1**
$\exists x\, (Jx \leftrightarrow Gx)$: **1**
$\forall x\, (Jx \leftrightarrow Gx)$: **0**
$\exists x\, ((Hx \vee Jx) \leftrightarrow Gx)$: **1**
$\forall x\, ((Hx \vee Jx) \leftrightarrow Gx)$: **0**
$\forall x\, ((Hx \vee Ix) \leftrightarrow {\sim}Jx)$: **1**
$\forall x\, ((Hx \vee Ix) \leftrightarrow Ix)$: **1**
$\forall x\, ((Hx \wedge Ix) \leftrightarrow Ix)$: **1**
$\exists x\, ((Hx \wedge Ix) \wedge {\sim}Gx)$: **1**

Summary: Instances and Quantifier Semantics

- **Instance of a Quantified Sentence**: an instance of a quantified sentence is the result of (i) removing the (left-most) quantifier from that sentence and (ii) replacing the variable of quantification (the variable appearing in that quantifier) in the scope formula, according to the following three constraints.

  (1) Replace **all free** occurrences of the variable.
  (2) Replace **only free** occurrences of the variable.
  (3) Replace occurrences of the variable by the **same name letter** throughout.

- **Existential Semantics**:

  An **existential** sentence is **true** in a model if (and only if) it has **at least one true instance** in that model.

  An **existential** sentence is **false** in a model if (and only if) **every instance** of that sentence is **false** in the model.

- **Universal Semantics**:

  A **universal sentence** is **true** in a model if (and only if) **every instance** of that sentence is **true** in the model.

  A **universal sentence** is **false** in a model if (and only if) it has **at least one false instance** in that model.